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Relativistic wavefunctions on spinor spaces

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Abstract. In extending the work done by Wigner in a 1963 paper, we introduce the Poincaré group representations of functions defined on the space of complex spinors. We give a geometrical interpretation of the spinor space for both the massive and massless cases. In the massive case we get a six-dimensional manifold, with three compact dimensions. In the massless case we get a four-dimensional manifold, with one compact dimension.

1. Introduction

In 1939 Wigner found the elementary building blocks of any quantum mechanics consistent with the Poincaré group (inhomogeneous, proper orthochronous Lorentz group) [1]. These building blocks are the irreducible unitary representations of the covering group of this group, which are related to elementary particles. The usual description of elementary particles is in terms of vector fields or spinor fields [2]. The connection between such fields and the irreducible representations was studied by several authors for particular cases [3]. This connection is not one to one; in particular, non-scalar tachyons and mass-zero particles of infinite spin ($(0, \Xi)$ irreducible representations, see below) cannot be described by tensor fields or spinor fields in Minkowski space [3, 4].

In 1963 Wigner wrote a paper on the connection between the irreducible representations and wavefunctions describing the elementary particles [5]. His discussion was limited to particles of integer spin, but it applied to massive particles, mass-zero particles [6] and particles of imaginary mass (tachyons). The description was in terms of complex-valued functions of two translationally invariant 4-vectors p (linear momentum) and q. These vectors are subject to the invariant constraints: p^2, q^2 and $p \cdot q$ fixed. Among the nice features of this approach to the integer spin representations of the Lorentz group is the intuitive geometrical interpretation that one can give to the action of the group itself on these representations in terms of the manifold described by p and q.

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In the massive case the action of the group is represented geometrically by the five-dimensional manifold $H \times S^2$ where H is the mass hyperboloid and the Lorentz group acts on the momentum variable p. The action of the little group on the variable q determines the spin of the representation, this motion being on the 2-sphere (we explain this further in the following).

In the massless case the mass hyperboloid is replaced by the light cone and the 2-sphere by a cylinder, with axis along p. Unitarity forces us to gauge out the translations along the surface of the cylinder, so that the manifold on which the Hilbert space is defined is actually the light cone with a circle at each point. We have a four-dimensional manifold.

In the present paper we extend Wigner's 1963 paper to half-integer spin. A straightforward extrapolation would lead one to expect complex-valued wavefunctions depending on a 4-vector p and on a spinor. We find, however, that the optimal description involves functions of two complex 2-spinors.

In the massive and imaginary mass case we consider functions $f(\eta, \xi, \bar{\eta}, \bar{\xi})$ of two complex 2-spinors (and their complex conjugates). In the massless case we consider functions $f(\eta, \bar{\eta})$ of just one spinor. A geometrical interpretation of the spinor space is given for both the massive and massless case and it is compared to the geometrical interpetation of the integer spin representations given by Wigner. We find that the massive case only requires a higher-dimensional space: a six-dimensional manifold. The approach through the spinor space is very natural and quite simple in the massless case.

In reviewing the integer spin massive case, we have slightly modified one of the constraints used by Wigner in a way that facilitates the calculations without losing contact with Wigner's theory. An analogous constraint can be used in the half-integer spin massive case, yielding an invariant differential equation for the half-integer spin particles.

The paper is organised as follows. In § 2 we introduce our spinor notation. In § 3 we consider the massive case. We first review Wigner's theory for the integer spin representations in terms of functions of p and q. Making the modifications just cited then enables us to easily derive the carrier spaces of the irreducible representations of the Poincaré group. We then consider in § 3.2 the half-integer representations on the functions on spinor space. In § 4 we consider the massless case. Here also we review Wigner's construction and then include the half-integer spin representations. In § 5 we discuss the case of imaginary mass. Finally in § 6 we give our conclusions.

2. Notation

Let η^A, for $A = 1, 2$, be a two-component spinor [3, 7]; namely, for $A \in \mathrm{SL}(2, C)$:

$$\eta^A \to \eta'^A = A^A{}_B \eta^B. \tag{2.1}$$

The complex conjugate of η^A is $\bar{\eta}^{\dot{A}}$, which transforms as

$$\bar{\eta}^{\dot{A}} \to \bar{\eta}'^{\dot{A}} = \bar{A}^{\dot{A}}{}_{\dot{B}} \bar{\eta}^{\dot{B}}. \tag{2.2}$$

We raise and lower indices using the 2×2 antisymmetric matrix ε:

$$\varepsilon_{AB} = \varepsilon^{AB} = \varepsilon_{\dot{A}\dot{B}} = \varepsilon^{\dot{A}\dot{B}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2.3}$$

and adopt the following conventions:

$$\eta^A = \eta_B \varepsilon^{BA} \qquad \bar{\eta}^{\dot{A}} = \bar{\eta}_{\dot{B}} \varepsilon^{\dot{B}\dot{A}} \tag{2.4}$$

$$\eta_A = \varepsilon_{AB} \eta^B \qquad \bar{\eta}_{\dot{A}} = \varepsilon_{\dot{A}\dot{B}} \bar{\eta}^{\dot{B}}. \tag{2.5}$$

Note that

$$A^t \varepsilon A = \varepsilon \tag{2.6}$$

from which it follows that, if η^A transforms according to A, $\bar{\eta}^{\dot{A}}$ transforms according to $\bar{A}$, η_A to $A^{t^{-1}}$ and $\bar{\eta}_{\dot{A}}$ to $A^{\dagger-1}$.

Furthermore we have that

$$(\eta \cdot \xi) \equiv \eta^A \xi_A = A^A{}_B A^{-1C}{}_A \eta^B \xi_C \tag{2.7}$$

is a Lorentz scalar.

Let us define the quantity

$$\bar{\eta}\sigma\xi \equiv \bar{\eta}^{\dot{A}} \xi^B \sigma_{\dot{A}B} \tag{2.8}$$

where σ stands for the vectorial notation of the Pauli matrices σ_μ, $\mu = 0, 1, 2, 3$ with indices $(\sigma_\mu)_{\dot{A}A}$.

We have that $\bar{\eta}\sigma\xi$ transforms as a 4-vector under the Lorentz group. Moreover, using the identity

$$(\sigma_\mu)_{A\dot{B}} (\sigma^\mu)_{C\dot{D}} = 2\varepsilon_{AC}\varepsilon_{\dot{B}\dot{D}} \tag{2.9}$$

we obtain the Fierz transformation:

$$\bar{\eta}\sigma\xi \cdot \bar{\alpha}\sigma\gamma = 2(\bar{\alpha} \cdot \bar{\eta})(\xi \cdot \gamma). \tag{2.10}$$

By abuse of notation, the dot product in (2.10) is between 4-vectors on the left-hand side, with metric $g_{\mu\nu}$ and between spinors on the right-hand side, with metric ε_{AB} or $\varepsilon_{\dot{A}\dot{B}}$. In fact, one can always write a 4-vector index as a pair of spinorial indices, one dotted and one undotted, and find [7] the following correspondences:

$$g_{\mu\nu} \leftrightarrow \varepsilon_{AB}\varepsilon_{\dot{A}\dot{B}} \tag{2.11}$$

$$\varepsilon_{\mu_1\mu_2\mu_3\mu_4} \leftrightarrow \tfrac{1}{4}\mathrm{i}(\varepsilon_{A_1A_4}\varepsilon_{A_2A_3}\varepsilon_{\dot{A}_1\dot{A}_3}\varepsilon_{\dot{A}_2\dot{A}_4} - \varepsilon_{A_1A_3}\varepsilon_{A_2A_4}\varepsilon_{\dot{A}_1\dot{A}_4}\varepsilon_{\dot{A}_2\dot{A}_3}). \tag{2.12}$$

Throughout the paper we shall denote 4-vectors by use of italic letters and 3-vectors by use of boldface letters.

3. Massive case

3.1. Integer spin

We briefly review the technique used by Wigner to describe the integer spin representations of the Lorentz group.

The carrier space is the space of complex-valued functions $f(p, q)$ depending on the momentum p and on another translationally invariant 4-vector q, satisfying the following constraints:

$$(p^2 - m^2) f(p, q) = 0 \tag{3.1}$$

$$p \cdot q f(p, q) = 0 \tag{3.2}$$

$$q^2 f(p, q) = -f(p, q). \tag{3.3'}$$

We find it more convenient, instead of (3.3′), to use the following equation:

$$\left[\frac{\partial}{\partial q}\cdot\frac{\partial}{\partial q}-\frac{1}{m^2}\left(p\cdot\frac{\partial}{\partial q}\right)^2\right]f(p,q)=0 \tag{3.3}$$

as we shall now justify.

Equation (3.3) means that in the rest frame of p, where $p=(m,0,0,0)$, $f(p,q)$ satisfies the Laplace equation $\Delta f(p,q)=0$. Instead of functions on the sphere $\boldsymbol{q}^2=+1$ in the rest frame [5], we have here solutions of Laplace's equation. If we put in the extra condition that we wish the solution to Laplace's equation to be regular at the origin $\boldsymbol{q}=\boldsymbol{0}$, then there is a one-to-one mapping between the solution to Laplace's equation and the spherical functions. This shows that the two constrains (3.3) and (3.3′) are interchangeable. We adopt (3.3) since it simplifies the solution of the eigenvalue problem for W^2 (see below).

The Lorentz group acts on the space of such functions by transforming, in the usual way, the two 4-vectors p and q. The invariant inner product (f,g) is given by

$$(f,g)=\int \mathrm{d}^4p\,\mathrm{d}^4q\,\delta(p^2-m^2)\delta(p\cdot q)\delta(q^2+1)\overline{f(p,q)}g(p,q). \tag{3.4}$$

In order to find the spin content of the representation, we calculate the Pauli-Lubanski 4-vector W:

$$W^\mu=\varepsilon^{\mu\alpha\beta\gamma}p_\alpha q_\beta\frac{\partial}{\partial q^\gamma}. \tag{3.5}$$

Making use of the following identity:

$$\begin{aligned}\varepsilon^{\mu\alpha\beta\gamma}\varepsilon_{\mu\alpha'\beta'\gamma'}=&-\delta^\alpha{}_{\alpha'}(\delta^\beta{}_{\beta'}\delta^\gamma{}_{\gamma'}-\delta^\gamma{}_{\beta'}\delta^\beta{}_{\gamma'})-\delta^\beta{}_{\alpha'}(\delta^\alpha{}_{\beta'}\delta^\gamma{}_{\gamma'}-\delta^\gamma{}_{\beta'}\delta^\alpha{}_{\gamma'})\\&-\delta^\gamma{}_{\alpha'}(\delta^\alpha{}_{\beta'}\delta^\beta{}_{\gamma'}-\delta^\beta{}_{\beta'}\delta^\alpha{}_{\gamma'})\end{aligned} \tag{3.6}$$

one finds from (3.1) and (3.2)

$$-W^\mu W_\mu=m^2S(S+1)-q^2\left[m^2\frac{\partial}{\partial q}\cdot\frac{\partial}{\partial q}-\left(p\cdot\frac{\partial}{\partial q}\right)^2\right] \tag{3.7}$$

where

$$S=q^\beta\frac{\partial}{\partial q^\beta}. \tag{3.8}$$

Thus, using (3.3) we have

$$W^2f(p,q)=-m^2S(S+1)f(p,q). \tag{3.9}$$

The solutions in the rest frame are then polynomials in $\boldsymbol{q}$, which satisfy Laplace's equation, as we explained above. This implies, in particular, traceless conditions on the tensors involved.

For a general p we can write the polynomial solutions for given eigenvalues of S straightforwardly:

$S=0 \qquad f(p,q)=f(p)$

$S=1 \qquad f(p,q)=f_\mu(p)q^\mu \qquad f_\mu(p)p^\mu=0$

$S=2 \qquad f(p,q)=f_{\mu\nu}(p)q^\mu q^\nu \qquad f_{\mu\nu}(p)p^\nu=0 \qquad f_\mu{}^\mu(p)=0 \qquad$ etc.

Because of the one-to-one correspondence between the functions on the sphere and the solutions to Laplace's equation regular at the origin, the inner products of these functions are given by the integral over the unit sphere in the rest frame of p. For general p this translates to

$$\int \mathrm{d}^4 q\, \delta(p \cdot q)\delta(q^2+1). \tag{3.10}$$

3.2. *Including half-integer spin*

Wigner's analysis left open the case of the double-valued representations of the Lorentz group. To include them we consider complex-valued functions φ of two spinors η, ξ and their complex conjugates $\bar{\eta}$, $\bar{\xi}$ subject to the following constraints:

$$\begin{aligned} \eta \cdot \xi &= m \\ \bar{\eta} \cdot \bar{\xi} &= m. \end{aligned} \tag{3.11}$$

From (3.11) and (2.10) follows

$$\left(\frac{\bar{\eta}\sigma\eta + \bar{\xi}\sigma\xi}{2}\right)^2 = (\bar{\eta} \cdot \bar{\xi})(\eta \cdot \xi) = m^2. \tag{3.12}$$

We therefore regard as 4-momentum the vector:

$$p = \frac{\bar{\eta}\sigma\eta + \bar{\xi}\sigma\xi}{2}. \tag{3.13}$$

The easiest form for the spinors η and ξ, in the rest frame, $p = (m, 0, 0, 0)$, due to the constraint (3.11), is

$$\eta = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sqrt{m} \qquad \xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sqrt{m}. \tag{3.14}$$

Using such spinors we obtain the tetrad:

$$\begin{aligned} \frac{\bar{\eta}\sigma\eta + \bar{\xi}\sigma\xi}{2} &= m(1, 0, 0, 0) & \frac{\bar{\eta}\sigma\eta - \bar{\xi}\sigma\xi}{2} &= m(0, 0, 0, 1) \\ \frac{\bar{\eta}\sigma\xi + \bar{\xi}\sigma\eta}{2} &= m(0, 1, 0, 0) & \frac{\bar{\eta}\sigma\xi - \bar{\xi}\sigma\eta}{2\mathrm{i}} &= m(0, 0, 1, 0). \end{aligned} \tag{3.15}$$

We define the Poincaré group† action on φ, corresponding to a translation by a and a SL(2, C) transformation A, by

$$U_{(a,A)}\varphi(\eta, \xi, \bar{\eta}, \bar{\xi}) = \exp\left(\mathrm{i}a \cdot \frac{(\bar{\eta}\sigma\eta + \bar{\xi}\sigma\xi)}{2}\right) \varphi(A^{-1}\eta, A^{-1}\xi, \bar{A}^{-1}\bar{\eta}, \bar{A}^{-1}\bar{\xi}). \tag{3.16}$$

We introduce the scalar product:

$$(\varphi, \psi) = \int \mathrm{d}\eta\, \mathrm{d}\bar{\eta}\, \mathrm{d}\xi\, \mathrm{d}\bar{\xi}\, \delta(\eta \cdot \xi - m)\delta(\bar{\eta} \cdot \bar{\xi} - m)\overline{\varphi(\eta, \xi, \bar{\eta}, \bar{\xi})}\psi(\eta, \xi, \bar{\eta}, \bar{\xi}) \tag{3.17}$$

and consider the Hilbert space of functions φ such that

$$(\varphi, \varphi) < \infty.$$

† By Poincaré group in the following we actually mean the covering group of the Poincaré group.

Manifestly the representation U defined by (3.16) is a unitary representation of the Poincaré group on this space.

In order to classify the representation we calculate the invariant operator W^2. We calculate it in the rest frame $p=(m,0,0,0)$.

The generators S_i of rotations are given by

$$-2\mathrm{i}S_i=(\sigma_i\eta)^A\frac{\partial}{\partial\eta^A}+(\sigma_i\xi)^A\frac{\partial}{\partial\xi^A}-(\bar\sigma_i\bar\eta)^{\dot A}\frac{\partial}{\partial\bar\eta^{\dot A}}-(\bar\sigma_i\bar\xi)^{\dot A}\frac{\partial}{\partial\bar\xi^{\dot A}}. \tag{3.18}$$

Notice that the generators S_i are skew-Hermitian, as expected, since we work with a unitary representation. It follows that, in the rest frame, the three non-vanishing components of the Pauli–Lubanski 4-vector W are

$$2\frac{W_i}{\mathrm{i}m}=\eta^B(\sigma_i)^A{}_B\frac{\partial}{\partial\eta^A}+\xi^B(\sigma_i)^A{}_B\frac{\partial}{\partial\xi^A}-\bar\eta^{\dot B}(\bar\sigma_i)^{\dot A}{}_{\dot B}\frac{\partial}{\partial\bar\eta^{\dot A}}-\bar\xi^{\dot B}(\bar\sigma_i)^{\dot A}{}_{\dot B}\frac{\partial}{\partial\bar\xi^{\dot A}}. \tag{3.19}$$

The calculation of W^2 follows from (3.19) straightforwardly, but the large number of terms that one gets does not suggest an easy interpretation of the differential operators which appear in W^2 nor a simple way of determining the eigenfunctions of W^2. A better way of approaching the problem is to pass from the space of functions of η, $\bar\eta$, ξ, $\bar\xi$ to one in which p appears explicitly as an independent variable. We notice that, because of (3.13), the following Dirac-type equations hold:

$$\begin{aligned}(p^\mu\sigma_\mu)_{\dot AA}\xi^A&=-(\eta\cdot\xi)\bar\eta_{\dot A}\\ (p^\mu\sigma_\mu)_{\dot AA}\eta^A&=+(\eta\cdot\xi)\bar\xi_{\dot A}.\end{aligned} \tag{3.20}$$

This means that $\bar\xi$ and $\bar\eta$ are, for a given p, not independent of ξ and η: for a fixed p, one only has a function of two spinors, subject to the constraints (3.11)†.

We can therefore make the following change of variables:

$$\eta,\bar\eta,\xi,\bar\xi\rightarrow p,\rho,\zeta \tag{3.21}$$

where η and ξ remain unchanged:

$$\eta=\rho\qquad\qquad \xi=\zeta. \tag{3.22}$$

$p(\eta,\xi,\bar\eta,\bar\xi)$ is given by (3.13) and the inverse transformations $\bar\eta(p,\rho,\zeta)$, $\bar\xi(p,\rho,\zeta)$ are given by (3.20).

The Jacobian of this coordinate transformation is easily evaluated and we get

$$\frac{\partial(p,\rho,\zeta)}{\partial(\eta,\xi,\bar\eta,\bar\xi)}=\tfrac14\mathrm{i}(\eta\cdot\xi)^2. \tag{3.23}$$

Hence the real measure on the space of η, ξ, $\bar\eta$, $\bar\xi$ can be expressed as follows:

$$\begin{aligned}&\mathrm{d}\eta\,\mathrm{d}\bar\eta\,\mathrm{d}\xi\,\mathrm{d}\bar\xi\,\delta(\eta\cdot\xi-m)\delta(\bar\eta\cdot\bar\xi-m)\\ &\quad=4\mathrm{d}^4p\,\frac{\mathrm{d}\eta\,\mathrm{d}\xi}{\mathrm{i}(\eta\cdot\xi)^2}\,\delta(\eta\cdot\xi-m)\delta\left(\frac{p^2}{\eta\cdot\xi}-m\right)\end{aligned} \tag{3.24}$$

where we have used the fact that $p^2=(\eta\cdot\xi)\,(\bar\eta\cdot\bar\xi)$, from (3.12).

Notice that by combining the two δ functions we obtain the invariant measure on the mass hyperboloid:

$$\mathrm{d}^4p\,\delta(p^2-m^2)\sim\mathrm{d}^3p/2p_0.$$

† If z is the complex number $z=z_0+\mathrm{i}z_1$, $z_0,z_1\in R$, z is considered to be independent of $\bar z$ just like z_0 is independent of z_1, namely in a real structure (we always count real parameters, the measures are real, etc).

The remaining part contains four formally real variables linked by one constraint. We get therefore a total of six parameters, one more than the number of parameters used by Wigner in the construction of the integer spin representations. The extra parameter—needed in order to build the half-integer spin representations—is due to the fact that, in constructing a fixed 4-vector out of two spinors and their complex conjugates, we can always change the spinors by a phase leaving the 4-vector invariant. This phase does not appear in the integer spin representations, as Wigner built them, out of the two 4-vectors p and q.

In order to show this explicitly let us switch from the set of variables η, $\bar{\eta}$, ξ, $\bar{\xi}$ to a new set of independent variables which includes the vector p as well as a spacelike 4-vector q.

Let us take for q the vector:

$$q=\frac{\bar{\eta}\sigma\eta-\bar{\xi}\sigma\xi}{2} \tag{3.25}$$

which satisfies

$$\begin{aligned} p\cdot q&=0\\ q^2&=-p^2. \end{aligned} \tag{3.26}$$

Because of the constraints (3.26) the number of degrees of freedom that q adds is only two, say q_1 and q_3. A convenient set of independent variables is the following:

$$\boldsymbol{p}, q_1, q_3, (\eta\cdot\xi), (\bar{\eta}\cdot\bar{\xi}), \psi.$$

Here we have introduced the new variables $(\eta\cdot\xi)$ and $(\bar{\eta}\cdot\bar{\xi})$, which may be removed by the δ functions that appear in (3.24). Further:

$$\psi=\tan^{-1}\mathrm{i}\frac{\bar{\eta}^1-\eta^1}{\bar{\eta}^1+\eta^1} \tag{3.27}$$

is the extra parameter that we get compared with Wigner's approach.

One can check that these new variables are independent and lead to a new integration measure:

$$\mathrm{d}\eta\,\mathrm{d}\bar{\eta}\,\mathrm{d}\xi\,\mathrm{d}\bar{\xi}=\mathrm{d}^3\boldsymbol{p}\,\mathrm{d}(\eta\cdot\xi)\,\mathrm{d}(\bar{\eta}\cdot\bar{\xi})\,\mathrm{d}q_1\,\mathrm{d}q_3\,\mathrm{d}\psi|J| \tag{3.28}$$

where J is the Jacobian of the transformation.

In order to understand the meaning of the parameter ψ and, in particular, why it is needed in the spinorial case, consider the fact that we can obviously change η by a phase ψ and ξ by a phase φ without affecting p and q; but in order to leave $\eta\cdot\xi$ and $\bar{\eta}\cdot\bar{\xi}$ unchanged we must have $\psi=-\varphi$. Therefore there is only one extra degree of freedom in the space of the spinors with respect to the space of the 4-vectors p and q.

Including the half-integer spin representations corresponds, from the point of view of the group, to going from the Lorentz group to its double cover SL(2, C). Accordingly, we can give a geometrical interpretation of the covering homomorphism by means of the parameter ψ.

The geometrical description of the Lorentz group, for massive (integer spin) representations, is that of a hyperboloid (the massive orbit) with a 2-sphere attached to each point. The sphere represents the rotation group, leaving a fixed point on the hyperboloid stable, in the space of functions of p and q. The eigenfunctions of W^2 corresponding to a fixed eigenvalue can be labelled to give the number of degrees of freedom of a fixed integer spin particle.

By including the parameter ψ, we are adding a circle to each point of the sphere on the mass hyperboloid. To see how this affects the representation let us calculate W^2. We shall consider that part of W^2 which involves ψ. If we make the calculation in the rest frame, $p=(m, 0, 0, 0)$, we need to know only the transformation of ψ under SU(2). It is possible to show, using the definition (3.27), that the (skew-Hermitian) generators of rotations S_x, S_y, S_z, in the space of functions of q_1, q_3, ψ restricted to ψ are

$$\begin{aligned} S_x &= \frac{r_2}{2r_1}\sin(\chi-\psi)\frac{\partial}{\partial\psi} \\ S_y &= \frac{r_2}{2r_1}\cos(\chi-\psi)\frac{\partial}{\partial\psi} \\ S_z &= \frac{1}{2}\frac{\partial}{\partial\psi} \end{aligned} \tag{3.29}$$

where†

$$r_2=(\bar\eta^2\eta^2)^{1/2} \qquad r_1=(\bar\eta^1\eta^1)^{1/2} \qquad \chi=\tan^{-1} \mathrm{i}\,\frac{\bar\eta^2-\eta^2}{\bar\eta^2+\eta^2}.$$

Since W^2 is an invariant we can consider the frame in which the component η^2 of the spinor η vanishes, and still $p=(m, 0, 0, 0)$. In this case S_x and S_y vanish and we find

$$W^2=-m^2\left(\frac{1}{2}\frac{\partial}{\partial\psi}\right)^2 \equiv m^2S^2 \qquad S=\frac{-\mathrm{i}}{2}\frac{\partial}{\partial\psi}. \tag{3.30}$$

The eigenfunctions of the operator S with eigenvalue s are

$$f(\psi)=\exp(\mathrm{i}2s\psi) \tag{3.31}$$

where $2s$ has to be an integer because of the definition of ψ. We see therefore that the introduction of ψ allows us to add half-integer spin representations to the integer spin ones defined by Wigner. We also get the usual integer spin representations corresponding to integer values of s in (3.31), but for a fixed s we get different, although equivalent, representations, as we shall discuss later.

To investigate on the content of our representation let us go back to the space of p, η, ξ‡, defined by equations (3.21)-(3.24). In the space the calculation of W^2 is easier since, having explicit expressions in p, we may calculate in the rest frame. We thus get

$$W^2=m^2[S(S+1)+V] \tag{3.32}$$

where

$$S=\tfrac{1}{2}\left(\eta\cdot\frac{\partial}{\partial\eta}+\xi\cdot\frac{\partial}{\partial\xi}\right)=\tfrac{1}{2}\left(\eta^A\frac{\partial}{\partial\eta^A}+\xi^B\frac{\partial}{\partial\xi^B}\right) \tag{3.33}$$

and

$$V=\eta\cdot\xi\frac{\partial}{\partial\eta}\cdot\frac{\partial}{\partial\xi}\equiv\eta^A\xi_A\frac{\partial}{\partial\eta_B}\frac{\partial}{\partial\xi^B}. \tag{3.34}$$

† The algebra of the operators (3.29) does not close because we are not considering the transformations of q_1 and q_3.

‡ We identify η with ρ and ξ with ζ, whenever this does not cause confusion.

Both operators S and V are explicitly invariant and we now show that both are Hermitian. In order to check this, one can just check the hermiticity of S, since W^2 is Hermitian by definition. If one works with the variables p, η, ξ, one has to rewrite the conjugate of the spinors in terms of these variables in order to find what the adjoint of S is. A better procedure is to write the expression of S in the space of variables η, ξ, $\bar{\eta}$, $\bar{\xi}$, since in this space one gets

$$S=\tfrac{1}{2}\left(\eta\cdot\frac{\partial}{\partial\eta}+\xi\cdot\frac{\partial}{\partial\xi}-\bar{\eta}\cdot\frac{\partial}{\partial\bar{\eta}}-\bar{\xi}\cdot\frac{\partial}{\partial\bar{\xi}}\right) \tag{3.35}$$

which is explicitly Hermitian†. Therefore we have proved that both operators S and V are indeed Hermitian.

The eigenfunctions of W^2 are in general difficult to analyse. Two particular simplifications, however, enable us to obtain explicit solutions (including those of Wigner). We now restrict our attention to two subspaces of our Hilbert space, on which either one of the operators V and S is a null operator.

(i) Let us consider the subspace of functions of the variables p, η, ξ which do not depend on ξ. This is a particular case of functions of our carrier space which satisfy the relativistic invariant equation $Vf=0$:

$$\eta\cdot\xi\frac{\partial}{\partial\eta}\cdot\frac{\partial}{\partial\xi}f(p,\eta,\xi)=0. \tag{3.36}$$

The eigenfunctions of S (and hence of W^2 for the case of vanishing V) are homogeneous polynomials in η and ξ. If they do not depend on ξ—and analogously if they did not depend on η—we get

$S=0$ $\quad f(p,\eta,\xi)=\varphi(p)$

$S=\tfrac{1}{2}$ $\quad f(p,\eta,\xi)=\psi_A(p)\eta^A$

$S=1$ $\quad f(p,\eta,\xi)=\chi_{AB}(p)\eta^A\eta^B$ $\quad \chi_{AB}$ symmetric in A and B $\quad$ etc.

The integer spin representations obtained by Wigner on the space of functions of p and q are not included in this case. We shall examine them now.

(ii) Let us consider the subspace of functions of the variables p, η, ξ which depend only on p and q, where q is the vector (3.25). In order to write q explicitly in terms of the variables p, η, ξ, we use equations (3.20) and find that

$$q^\nu=\tau^\nu_{CB}\frac{\xi^C\eta^B}{\eta\cdot\xi}\qquad \tau^\nu_{CB}\equiv\tfrac{1}{2}\varepsilon^{\dot{A}\dot{B}}p^\mu(\sigma_{\mu\dot{B}C}\sigma^\nu_{\dot{A}B}+\sigma_{\mu\dot{B}B}\sigma^\nu_{\dot{A}C}). \tag{3.37}$$

The operator S identically vanishes when acting on q because the operators $\eta\cdot\partial/\partial\eta$ and $\xi\cdot\partial/\partial\xi$ separately vanish on q. In fact

$$\eta^A\frac{\partial}{\partial\eta^A}q^\nu=\tau^\nu_{CB}\frac{\eta^A\xi^C\delta^B_A(\eta\cdot\xi)-\eta^A\xi^C\eta^B\xi_A}{(\eta\cdot\xi)^2}=0 \tag{3.38}$$

and analogously for $\xi\cdot\partial/\partial\xi$.

† The space of η, $\bar{\eta}$, ξ, $\bar{\xi}$ is much easier to work with for what concerns the reality and invariance of the integration measure and the hermiticity of the operators. In this respect it is the most natural space for the representations of SL(2, C) acting on a space of complex-valued functions.

Therefore the action of the operator W^2 on the functions defined only on the variables p and q is just the action of the operator V. Since V can also be written in the following way:

$$V=\eta^A\xi_A\frac{\partial}{\partial\eta_B}\frac{\partial}{\partial\xi^B}=2+\eta^A\frac{\partial}{\partial\eta^A}+\xi^A\frac{\partial}{\partial\xi^A}+\frac{\partial}{\partial\eta_A}\frac{\partial}{\partial\xi^A}\eta\cdot\xi \tag{3.39}$$

we can use (3.38) and the analogue for $\xi\cdot\partial/\partial\xi$, and write

$$Vf(q)=\left(2+\frac{\partial}{\partial\eta_A}\frac{\partial}{\partial\xi^A}\eta\cdot\xi\right)f(q). \tag{3.40}$$

Let us consider the action of the operator V on the polynomials in q:

$$\begin{aligned}
V(q^{\nu_1}\dots q^{\nu_s})&=\left(2+\frac{\partial}{\partial\eta_A}\frac{\partial}{\partial\xi^A}(\eta\cdot\xi)^{1-s}(\eta\cdot\xi)^s\right)q^{\nu_1}\dots q^{\nu_s}\\
&=\left(2+\frac{\partial}{\partial\eta_A}(s-1)\eta_A+\frac{\partial}{\partial\eta_A}(\eta\cdot\xi)^{1-s}\frac{\partial}{\partial\xi^A}(\eta\cdot\xi)^s\right)q^{\nu_1}\dots q^{\nu_s}\\
&=\Bigg(2+2(s-1)-(1-s)(\eta\cdot\xi)^{-s}\xi^A\frac{\partial}{\partial\xi^A}(\eta\cdot\xi)^s\\
&\quad+(\eta\cdot\xi)^{1-s}\frac{\partial}{\partial\eta_A}\frac{\partial}{\partial\xi^A}(\eta\cdot\xi)^s\Bigg)q^{\nu_1}\dots q^{\nu_s}\\
&=s(s+1)q^{\nu_1}\dots q^{\nu_s}+\sum_{i=1}^{s}\sum_{j>i}2(p^2g^{\nu_i\nu_j}-p^{\nu_i\nu_j})\prod_{k\neq i,k\neq j}q^{\nu_k}
\end{aligned} \tag{3.41}$$

where $g^{\mu\nu}$ is the Minkowski metric tensor.

We consider functions $f_{\nu_1\dots\nu_s}(p)q^{\nu_1}\dots q^{\nu_s}$ satisfying

$$f_{\nu_1\dots\nu_s}(p^{\nu_i}p^{\nu_j}-m^2g^{\nu_i\nu_j})=0.$$

These functions can also be considered transverse to p, since q is transverse (see (3.37)). The new condition then implies that they also be traceless.

We thus find the same result as in the integer spin case, that the eigenfunctions of W^2 with eigenvalue $s(s+1)$ are functions of p times a polynomial in q of order s:

$S=0$	$f(p,q)=f(p)$			
$S=1$	$f(p,q)=f_\mu(p)q^\mu$	$f_\mu(p)p^\mu=0$		
$S=2$	$f(p,q)=f_{\mu\nu}(p)q^\mu q^\nu$	$f_{\mu\nu}(p)p^\nu=0$	$f_\mu{}^\mu(p)=0$	etc.

These are the functions found by Wigner.

As we already mentioned before, we get more than one realisation of the representation corresponding to a fixed S, as explicitly shown in cases (i) and (ii) above for $S=1$. We can easily make an analogy with the covariant representations, namely the representation U^D having, as carrier space, the space of functions of p taking values in the carrier space of the representation D of SL(2, C). The integer spin representation studied by Wigner correspond to the representations $U^{D^{n,n}}$; the representations we add, through the parameter ψ, correspond to the representations $U^{D^{l,m}}$, with l, m, n integer or half-integer. The representations $U^{D^{n,n}}$ and $U^{D^{l,m}}$ are equivalent whenever $2n=l+m$.

For $S=1$ the representation found in case (i) corresponds to the representation $U^{D^{1,0}}$ and the one found in case (ii) to the $U^{D^{\frac{1}{2},\frac{1}{2}}}$. The representations $U^{D^{1,0}}$ and $U^{D^{\frac{1}{2},\frac{1}{2}}}$ are equivalent since both correspond to the representation induced by D^1 of SU(2). Notice that the equivalence of our representations, in the case $S=1$, is quite obvious if we write q explicitly in terms of η and ξ, using (3.37). In fact

$$f_\nu(p)q^\nu = f_\nu(p)\tau^\nu_{CB}\frac{\xi^C\eta^B}{\eta\cdot\xi} \equiv \kappa_{CB}(p)\xi^C\eta^B \tag{3.42}$$

where $\kappa_{CB}(p)$ is symmetric in C and B just like χ_{AB} in case (i). The equivalence of the representations can be easily shown through the correspondence between κ and χ.

4. Massless case

4.1. Integer helicity

Again we will begin by reviewing Wigner's technique for describing the integer helicity massless representations before extending it to include half-integer helicities. In particular, Wigner focuses [5] his attention on the infinite spin representations, which do not correspond to states of particles having a fixed integer helicity, since the eigenvalue of the angular momentum operator can extend to infinity. These representations, in fact, once restricted to the little group, are not one-dimensional unitary representations, as in the case of the representations of fixed (finite) helicity, but infinite-dimensional unitary representations.

The values of the invariant operators P^2 and W^2, in an infinite spin representation are 0 and Ξ^2, respectively, for $\Xi\neq 0$. Hence they are denoted as $0(\Xi)$. In the case of the representations of finite helicity, these eigenvalues are instead both 0.

In order to build the $0(\Xi)$ representation let us consider, as in the massive case, the space of complex-valued functions $f(p, q)$ depending on the momentum p and on another 4-vector q. We now impose the following constraints†:

$$p^2 f(p, q) = 0 \tag{4.1}$$

$$p\cdot q f(p, q) = 0 \tag{4.2}$$

$$q^2 f(p, q) = -f(p, q). \tag{4.3}$$

The Lorentz group acts on the space of such functions by transforming the two 4-vectors p and q. In order to find the helicity content of the representation, we calculate the Pauli–Lubanski 4-vector W. We find, as in the massive case,

$$W^\mu = \varepsilon^{\mu\alpha\beta\gamma}p_\alpha q_\beta\frac{\partial}{\partial q^\gamma} \tag{4.4}$$

but, this time, W^2 has a simpler expression, since we get

$$W^\mu W_\mu = S^2 \qquad \text{where} \qquad S = \mathrm{i}p^\mu\frac{\partial}{\partial q^\mu}. \tag{4.5}$$

One therefore obtains the $0(\Xi)$ representation through the equation:

$$Sf(p, q) = \mathrm{i}p^\mu\frac{\partial}{\partial q^\mu}f(p, q) = \Xi f(p, q). \tag{4.6}$$

† Because of these constraints, the measure with respect to which the function f is square integrable is formally written as $\mathrm{d}^4q\,\mathrm{d}^4p\,\delta(p^2)\delta(p\cdot q)\,\delta(q^2+1)$.

In the rest frame, namely for $p=(p^0, 0, 0, p^0)$, the constraints on p and q confine q to the surface of a cylinder, whose axis is along p. Therefore f is constrained on the surface of a cylinder, moved around by a Lorentz transformation. An analogous analysis can be made for the finite helicity representations.

4.2. Including half-integer helicity

The massless representations of SL(2, C) on the space of functions of spinors are extremely simple and make one really appreciate the advantage of working on such spaces.

We consider complex-valued functions φ of just one spinor η and its complex conjugate $\bar{\eta}$.

Since $(\bar{\eta}\sigma\eta)^2=0$, we regard $p=(\bar{\eta}\sigma\eta)$ as 4-momentum.

We define the Poincaré group action on φ, corresponding to a translation by a and a SL(2, C) transformation A, is given by

$$U_{(a,A)}\varphi(\eta, \bar{\eta}) = \exp(\mathrm{i}a\cdot\bar{\eta}\sigma\eta)\varphi(A^{-1}\eta, \bar{A}^{-1}\bar{\eta}). \tag{4.7}$$

We introduce the scalar product:

$$(\varphi, \psi) = \int \mathrm{d}\eta\, \mathrm{d}\bar{\eta}\, \bar{\varphi}(\eta, \bar{\eta})\psi(\eta, \bar{\eta}). \tag{4.8}$$

Again the Hilbert space of functions φ is taken to consist of those functions φ such that

$$(\varphi, \varphi) < \infty.$$

One may verify that the representation U defined by (4.7) is a unitary representation of the Poincaré group.

Since no invariant can be made out of η and $\bar{\eta}$ we cannot impose any constraint.

In the frame $p=(1, 0, 0, 1)$ we have

$$\eta = \begin{pmatrix} \exp(\mathrm{i}\psi) \\ 0 \end{pmatrix} \tag{4.9}$$

where ψ is a free parameter. Because of (4.9) the four components of p are not independent functions of η and $\bar{\eta}$. Hence we cannot transform the measure $\mathrm{d}\eta\,\mathrm{d}\bar{\eta}$ into the measure d^4p. We can however define, as for the massive case,

$$\psi = \tan^{-1} \mathrm{i}\frac{\bar{\eta}^1-\eta^1}{\bar{\eta}^1+\eta^1} \tag{4.10}$$

and find that the Jacobian of the transformation from the space of η, $\bar{\eta}$ to the space of $\boldsymbol{p}$ (3-vector) and ψ does not vanish. We get, in fact

$$\mathrm{d}\eta\,\mathrm{d}\bar{\eta} = \frac{1}{2|\boldsymbol{p}|}\mathrm{d}^3\boldsymbol{p}\,\mathrm{d}\psi. \tag{4.11}$$

On the RHS of (4.11) we find precisely the invariant measure on the light cone times the differential of the parameter ψ.

In the frame $p=(1, 0, 0, 1)$ (and so η given by (4.9)) the SL(2, C) action on ψ reduces to a rotation S_z about the z axis. This rotation is also an element of the stability group of p.

In the massless case W^2 is the sum of the squares of the two translation generators of the little group, calculated in the frame $p=(1,0,0,1)$. A straightforward evaluation of these generators shows that $W^2=0$ in our representation. Further

$$S_z=\frac{-\mathrm{i}}{2}\frac{\partial}{\partial\psi} \tag{4.12}$$

assumes the meaning of helicity operator (for the expression of S_z see § 3.2), with eigenfunctions

$$f(p,\psi)=g(p)\exp(\mathrm{i}2s\psi) \qquad 2s=0,1,2,\ldots. \tag{4.13}$$

If we consider the geometrical interpretation of the massless representations, as we have done in the massive case, we see that there is no difference here between the integer and half-integer representations. In both cases we just need an extra parameter, representing the rotations around the z axis, to get the helicity states. This should cause no surprise: the number of degrees of freedom for a massless particle is 1, no matter what its helicity is. Analogous to the massive case where the little group acted on a 2-sphere we now consider action on a cylinder whose axis lies along the momentum p. The only symmetry which counts, though, is the rotation of the cylinder around the z axis, since the translations by a vector proportional to p are to be gauged out because of unitarity (they would lead to null norm vectors in the Hilbert space). In our representations we just count the minimal number of parameters needed in order to describe a massless particle and no gauge condition has to be imposed.

Going back to the space of η and $\bar\eta$, which is a most natural space to work with in the massless case, we can calculate W^2 directly and we find, in the frame $p=(1,0,0,1)$

$$W^2=S\bar S \qquad \text{where} \qquad S=\eta^2\frac{\partial}{\partial\eta^1}. \tag{4.14}$$

Since in this frame $\eta^2=0$, it follows that $W^2=0$. The expression for S_z is

$$S_z=\tfrac12\left(\eta^1\frac{\partial}{\partial\eta^1}-\eta^2\frac{\partial}{\partial\eta^2}-\bar\eta^{\dot 1}\frac{\partial}{\partial\bar\eta^{\dot 1}}+\bar\eta^{\dot 2}\frac{\partial}{\partial\bar\eta^{\dot 2}}\right). \tag{4.15}$$

Hence the helicity states obviously are

$S=0$ $\qquad \varphi(\eta,\bar\eta)=\varphi(\bar\eta\sigma\eta)$

$S=\frac12$ $\qquad \varphi(\eta,\bar\eta)=\varphi_A(\bar\eta\sigma\eta)\eta^A$

$S=1$ $\qquad \varphi(\eta,\bar\eta)=\varphi_{AB}(\bar\eta\sigma\eta)\eta^A\eta^B \qquad \varphi_{AB}$ symmetric in A and B $\qquad$ etc.

The case in which the carrier space of the representation is a space of functions of η and $\bar\eta$ is, as we have shown, the simplest and the most natural to work with, in order to get the finite helicity representations of the Poincaré group. We could have worked, however, with functions of η, ξ, $\bar\eta$, $\bar\xi$ as well, in analogy with the massive case, and impose the constraint $\eta\cdot\xi=0$. We would have found, in particular, the infinite spin representations described by Wigner. We showed only the present formulation because of its simplicity and elegance.

5. Imaginary mass case

5.1. Integer spin

We now examine the case in which $p^2=-m^2$, for real m; as customary we call this the imaginary mass case. We will also speak of spin in the present case, even though this is not quite appropriate. The operator W^2, in fact, is no longer related to the angular momentum generators, as in the massive case, but rather to one generator of rotations and two generators of boosts. Nevertheless the eigenvalues of W^2 are still of the form $-m^2[s(s+1)]$ and we give such s the name of spin.

We consider the representation of the Poincaré group acting on the space of complex-valued functions $f(p, q)$ depending on the momentum p and on another translationally invariant 4-vector q, satisfying the following constraints:

$$(p^2+m^2)f(p,q)=0 \tag{5.1}$$

$$p\cdot qf(p,q)=0 \tag{5.2}$$

$$q^2f(p,q)=0. \tag{5.3}$$

Notice that in the present case we can have a vector q satisfying both $p\cdot q=0$ and $q^2=0$, which turns out to be a very convenient choice. In the massive case this was not possible, since a vector orthogonal to a timelike vector cannot be light-like.

The Lorentz group acts on the space of such functions by transforming, in the usual way, the two 4-vectors p and q. The inner product (f, g) is given by

$$(f,g)=\int \mathrm{d}^4p\,\mathrm{d}^4q\,\delta(p^2+m^2)\delta(p\cdot q)\delta(q^2)\overline{f(p,q)}g(p,q). \tag{5.4}$$

The calculations of the invariant W^2 using equations (3.5) and (3.6) is straightforward and the result is simpler than in the massive case because $q^2=0$. One finds, from (5.1)-(5.3),

$$-W^\mu W_\mu=m^2S(S+1) \qquad S=q^\beta\frac{\partial}{\partial q^\beta}. \tag{5.5}$$

The eigenfunctions of the operator W^2 are the eigenfunctions of the operator S. Hence they are polynomials in q times a function of p.

For a given eigenvalue of S we thus get

$S=0 \qquad f(p,q)=f(p)$

$S=1 \qquad f(p,q)=f_\mu(p)q^\mu \qquad f_\mu(p)p^\mu=0$

$S=2 \qquad f(p,q)=f_{\mu\nu}(p)q^\mu q^\nu \qquad f_{\mu\nu}(p)p^\nu=0 \qquad f_\mu{}^\mu(p)=0 \qquad$ etc.

These eigenfunctions are not in the Hilbert space carrying our representation: they are not normalisable in the inner product (5.4), since the integration is over the non-compact manifold spanned by q. This means that the values indicated above are in the continuous spectrum. There is no eigenfunction in the Hilbert space but there exist approximate eigenfunctions f such that, given $\varepsilon>0$, $\|Vf-s(s+1)f\|<\varepsilon\|f\|$. Such approximate eigenfunctions can be considered like the wavepackets, which are approximate eigenfunctions of the momentum operator in configuration space.

5.2. Including half-integer spin

We now include half-integer spin† representations for the imaginary mass case. We consider complex-valued functions φ of two spinors η, ξ and their complex conjugate $\bar{\eta}$, $\bar{\xi}$ subject to the following constraints:

$$\begin{aligned} \eta \cdot \xi &= m \\ \bar{\eta} \cdot \bar{\xi} &= m. \end{aligned} \tag{5.6}$$

From (5.6) and (2.10) it follows that

$$\left(\frac{\bar{\eta}\sigma\eta - \bar{\xi}\sigma\xi}{2}\right)^2 = -(\bar{\eta} \cdot \bar{\xi})(\eta \cdot \xi) = -m^2. \tag{5.7}$$

We therefore regard as 4-momentum the vector:

$$p = \frac{\bar{\eta}\sigma\eta - \bar{\xi}\sigma\xi}{2}. \tag{5.8}$$

When η and ξ, in a particular frame, assume the values

$$\eta = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sqrt{m} \qquad \xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sqrt{m} \tag{5.9}$$

then $p = (0, 0, 0, m)$, due to the constraint (5.6).

We define the Poincaré group action on φ, corresponding to a translation by a and a SL(2, C) transformation A, by

$$U_{(a,A)}\varphi(\eta, \xi, \bar{\eta}, \bar{\xi}) = \exp\left(\mathrm{i}a \cdot \frac{(\bar{\eta}\sigma\eta - \bar{\xi}\sigma\xi)}{2}\right)\varphi(A^{-1}\eta, A^{-1}\xi, \bar{A}^{-1}\bar{\eta}, \bar{A}^{-1}\bar{\xi}). \tag{5.10}$$

We introduce the scalar product:

$$(\varphi, \psi) = \int \mathrm{d}\eta\, \mathrm{d}\bar{\eta}\, \mathrm{d}\xi\, \mathrm{d}\bar{\xi}\; \delta(\eta \cdot \xi - m)\delta(\bar{\eta} \cdot \bar{\xi} - m)\overline{\varphi(\eta, \xi, \bar{\eta}, \bar{\xi})}\psi(\eta, \xi, \bar{\eta}, \bar{\xi}). \tag{5.11}$$

We define the Hilbert space of functions φ such that

$$(\varphi, \varphi) < \infty.$$

Again the representation U defined by (5.10) is a unitary representation of the Poincaré group.

In order to classify the representation we calculate the invariant W^2. We proceed the same way as we did in the massive case. First, we make the following change of variables:

$$\eta, \bar{\eta}, \xi, \bar{\xi} \to p, \eta, \xi. \tag{5.12}$$

We then calculate W^2 in the frame in which $p = (0, 0, 0, m)$. In this rest frame, the three non-vanishing components of the Pauli–Lubanski 4-vector W are

$$\begin{aligned} W_0 &= -m\left((\tfrac{1}{2}\mathrm{i}\sigma_z\eta)^A \frac{\partial}{\partial\eta^A} + (\tfrac{1}{2}\mathrm{i}\sigma_z\xi)^A \frac{\partial}{\partial\xi^A}\right) \\ W_1 &= -m\left((-\tfrac{1}{2}\sigma_y\eta)^A \frac{\partial}{\partial\eta^A} + (-\tfrac{1}{2}\sigma_y\xi)^A \frac{\partial}{\partial\xi^A}\right) \\ W_2 &= -m\left((\tfrac{1}{2}\sigma_z\eta)^A \frac{\partial}{\partial\eta^A} + (\tfrac{1}{2}\sigma_x\xi)^A \frac{\partial}{\partial\xi^A}\right). \end{aligned} \tag{5.13}$$

† Namely half-integer s such that the eigenvalue of the operator W^2 is $-m^2\,[s(s+1)]$.

Notice that, in the frame in which $p=(0,0,0,m)$, the Pauli–Lubanski 4-vector is written in terms of two boost generators (in the x and y directions) and the generator of the rotations around the z axis. We can therefore easily compare the Pauli–Lubanski vectors in the present case and in the massive case. It turns out that, in terms of the components appearing in equations (5.13), the Pauli–Lubanski vector in the imaginary mass case is $W=(W^0, W^1, W^2, 0)$ while in the massive case it is $W=(0, \mathrm{i}W^2, -\mathrm{i}W^1, W^0)$. We thus find that W^2, in the two cases, differs only by a sign. Therefore

$$W^2=-m^2[S(S+1)+V] \tag{5.14}$$

where

$$S=\tfrac{1}{2}\left(\eta\cdot\frac{\partial}{\partial\eta}+\xi\cdot\frac{\partial}{\partial\xi}\right)=\tfrac{1}{2}\left(\eta^A\frac{\partial}{\partial\eta^A}+\xi^B\frac{\partial}{\partial\xi^B}\right) \tag{5.15}$$

and

$$V=\eta\cdot\xi\frac{\partial}{\partial\eta}\cdot\frac{\partial}{\partial\xi}\equiv\eta^A\xi_A\frac{\partial}{\partial\eta_B}\frac{\partial}{\partial\xi^B}. \tag{5.16}$$

Both operators S and V are explicitly invariant and both are Hermitian, as we have shown in the massive case. Proceeding as we did in the massive case and restricting our attention to subspaces of our Hilbert space, for which either one of the operators V and S is a null operator, we find simple forms for the eigenfunctions of W^2.

(*a*) First, consider the subspace of functions of the variables p, η, ξ which do not depend on ξ. The eigenfunctions of W^2 are therefore homogeneous polynomials in η. Hence we obtain

$S=0$ $\quad f(p,\eta,\xi)=\varphi(p)$

$S=\tfrac{1}{2}$ $\quad f(p,\eta,\xi)=\psi_A(p)\eta^A$

$S=1$ $\quad f(p,\eta,\xi)=\chi_{AB}(p)\eta^A\eta^B$ $\quad \chi_{AB}$ symmetric in A and B $\quad$ etc.

The integer spin representations that we found in the preceding subsection, which act on the space of functions of p and q only, are not included in this case. We shall examine them now.

(*b*) Let us consider the subspace of functions of the variables p, η, ξ which depend only on p and q, where q is the vector:

$$q=\tfrac{1}{2}(\bar\eta+\bar\xi)\sigma(\eta+\xi) \tag{5.17}$$

which satisfies both

$$p\cdot q=0 \qquad\text{and}\qquad q^2=0. \tag{5.18}$$

In order to write q explicitly in terms of the variables p, η, ξ, we use the following equation:

$$p^\mu\sigma_{\mu\dot{A}B}(\eta+\xi)^B=-(\eta\cdot\xi)(\bar\eta+\bar\xi)_{\dot{A}}. \tag{5.19}$$

We thus find

$$q^\nu=\pi^\nu_{CB}\frac{(\eta+\xi)^C(\eta+\xi)^B}{\eta\cdot\xi} \qquad \pi^\nu_{CB}\equiv\tfrac{1}{2}\varepsilon^{\dot{A}\dot{B}}p^\mu\sigma_{\mu\dot{B}C}\sigma^\nu_{\dot{A}B}. \tag{5.20}$$

The operator S vanishes identically when acting on q, even though the operators $\eta\cdot\partial/\partial\eta$ and $\xi\cdot\partial/\partial\xi$ do not separately vanish on q. Therefore the action of the operator W^2 on functions depending only on the variables p and q is given just by the action of V. Moreover, since V can also be written in the following way:

$$V=\eta^A\xi_A\frac{\partial}{\partial\eta_B}\frac{\partial}{\partial\xi^B}=2+\eta^A\frac{\partial}{\partial\eta^A}+\xi^A\frac{\partial}{\partial\xi^A}+\frac{\partial}{\partial\eta_A}\frac{\partial}{\partial\xi^A}\eta\cdot\xi \tag{5.21}$$

and since S vanishes, when acting on functions of q only, we get

$$Vf(q)=\left(2+\frac{\partial}{\partial\eta_A}\frac{\partial}{\partial\xi^A}\eta\cdot\xi\right)f(q). \tag{5.22}$$

Let us now consider the action of the operator V on polynomials in q. The calculation is easier if we make the following change of variables:

$$\eta,\xi\to\alpha=\eta+\xi,\ \beta=\eta-\xi. \tag{5.23}$$

We have that

$$\begin{aligned}
&\alpha\cdot\beta=-\eta\cdot\xi\\
&\frac{\partial}{\partial\alpha_A}\frac{\partial}{\partial\beta^A}\alpha\cdot\beta=\frac{\partial}{\partial\eta_A}\frac{\partial}{\partial\xi^A}\eta\cdot\xi\\
&\frac{\partial}{\partial\beta^A}\alpha\cdot\beta q^\nu=0\\
&\alpha_A\frac{\partial}{\partial\alpha_A}q^\nu=q^\nu.
\end{aligned} \tag{5.24}$$

From (5.24) it follows that

$$\frac{\partial}{\partial\beta^A}(\alpha\cdot\beta)^s q^{\nu_1}\ldots q^{\nu_s}=0 \tag{5.25}$$

and

$$\begin{aligned}
Vq^{\nu_1}\ldots q^{\nu_s}&=\left(2+\frac{\partial}{\partial\alpha_A}\frac{\partial}{\partial\beta^A}\alpha\cdot\beta\right)q^{\nu_1}\ldots q^{\nu_s}\\
&=\left(2+\frac{\partial}{\partial\alpha_A}\frac{\partial}{\partial\beta^A}(\alpha\cdot\beta)^{1-s}(\alpha\cdot\beta)^s\right)q^{\nu_1}\ldots q^{\nu_s}\\
&=\left(2+\frac{\partial}{\partial\alpha_A}(s-1)\alpha_A\right)q^{\nu_1}\ldots q^{\nu_s}\\
&=\left(2+2(s-1)+(s-1)\alpha_A\frac{\partial}{\partial\alpha_A}\right)q^{\nu_1}\ldots q^{\nu_s}\\
&=s(s+1)q^{\nu_1}\ldots q^{\nu_s}.
\end{aligned} \tag{5.26}$$

The eigenfunctions of W^2 with eigenvalue $s(s+1)$ are functions of p times a polynomial in q of order s:

$$\begin{aligned}
&S=0 \qquad f(p,q)=f(p)\\
&S=1 \qquad f(p,q)=f_\mu(p)q^\mu\\
&S=2 \qquad f(p,q)=f_{\mu\nu}(p)q^\mu q^\nu \qquad \text{etc.}
\end{aligned}$$

We thus find the same result as in the integer spin case.

6. Conclusions

We have found here a uniform description giving wavefunctions for free particles belonging to all irreducible representations of the covering to the Poincaré group.

The usual description in terms of spinor or tensor fields for the massive case can be readily recovered from our formalism by considering functions which are finite polynomials in a subset of our variables. This suggests that an alternative approach to ours could be followed using Grassmann variables [8]. We are currently investigating such an approach.

For the massless case, except for simple helicity states, there is no tensor formulation. By contrast, our uniform approach works just as easily for the massless as well as for the massive case. Similarly, aside from scalar tachyons, there is no spinor field representation for tachyons†.

An important question to investigate is how to construct interactions in this formalism. Such a discussion is essential if one wishes to argue seriously about the existence of non-scalar tachyons.

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† Regarding the question of the existence of theories with interacting tachyons, a preprint [9] has recently appeared which demonstrates that theories with tachyons always fail to be unitary at one quantum loop and beyond.